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Engineering entrainment and adaptation in limit cycle systems

From biological inspiration to applications in robotics

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Abstract Periodic behavior is key to life and is observed in multiple instances and at multiple time scales in our metabolism, our natural environment, and our engineered environment. A natural way of modeling or generating periodic behavior is done by using oscillators, i.e., dynamical systems that exhibit limit cycle behavior. While there is extensive literature on methods to *analyze* such dynamical systems, much less work has been done on methods to *synthesize* an oscillator to exhibit some specific desired characteristics. The goal of this article is twofold: (1) to provide a framework for characterizing and designing oscillators and (2) to review how classes of well-known oscillators can be understood and related to this framework. The basis of the framework is to characterize oscillators in terms of their fundamental temporal and spatial behavior and in terms of properties that these two behaviors can be designed to exhibit. This focus on fundamental properties is important because it allows us to systematically compare a large variety of oscillators that might at first sight appear very different from each other. We identify several specifications that are useful for design, such as frequency-locking behavior, phase-locking behavior, and specific output signal shape. We also identify two classes of design methods by which these specifications can be met, namely offline methods and online methods. By relating these specifications to our framework and by presenting several examples of how oscillators have been designed in the literature, this article provides

a useful methodology and toolbox for designing oscillators for a wide range of purposes. In particular, the focus on synthesis of limit cycle dynamical systems should be useful both for engineering and for computational modeling of physical or biological phenomena.

1 Introduction

Periodic behavior is central to our lives. Our body functions thanks to many types of periodic behaviors ranging from heart beats, breathing, chewing, locomotion, and various rhythms in the brain to cycles in gene regulatory networks. Similarly our natural environments have multiple periodic phenomena such as rotations of the earth around the sun, seasons, tides, cycles in ecological systems (e.g., prey–predator populations), in chemical reactions, etc. Finally, many systems that we engineer are meant to exhibit periodic behavior such as clocks (for watches or CPUs), lasers, music, traffic lights, and satellites, to name just a few examples. All these phenomena share many common features and can be modeled (or controlled for engineered systems) by systems of differential equations that exhibit limit cycle behavior, that is, by oscillators.

The importance and ubiquity of periodic behavior explains why oscillator models are published in such a large variety of journals in different fields (nonlinear dynamics, physics, biology, chemistry, engineering, etc.). This makes oscillators a very exciting topic of study but at the same time makes it difficult to extract common principles from all these models. Indeed each field has its own terminology, variable/parameter names, systems of coordinates, methods of analysis, and methods of synthesis, and this makes it hard to see similarities and differ-

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ences between models. Furthermore, the choice of a particular oscillator in a given field is often not transparent and depends sometimes more on historical reasons than on pure design or modeling considerations. The reason for this is usually a lack of abstraction of the concept of oscillators. Finally, another difficulty with oscillators is that while there are many tools for analyzing the behavior of an oscillator (see Kuramoto 1984; Winfree 2001; Glass and Mackey 1988; Pikovsky et al. 2001 for some outstanding textbooks), there is a lack of methodologies for *designing* them to exhibit a particular behavior.

A goal of this article is therefore to provide a framework for characterizing different oscillator models in a systematic way and to focus on methodologies that can be used for designing them. In this process, we review a large range of oscillator models that have been developed as well as some of our own work in adaptive frequency oscillators. We will try to focus on the fundamental principles of limit cycle systems and separate those from unnecessary details of a particular implementation. These principles can then be applied to the *design* of systems and used to provide guidelines on how to endow a system with a set of predefined properties. Our perspective is therefore mainly an engineering one, i.e., we want to address the problem of how an oscillator or a system of coupled oscillators can be designed to do something useful (e.g., for coordination, sequencing, and/or pattern formation), but the approach should also be interesting for computational modeling.

Oscillators are of interest for engineers for several reasons. They can be exploited for timing and sequencing. They can synchronize to external signals and show coordinated behavior with perturbations and other oscillators. Connecting them into networks or lattices, they can form coordinated yet flexible spatiotemporal patterns. These networks can act as pattern generators that can reduce the dimensionality of a given control problem, in the sense that a small number of simple (scalar) parameters can control multidimensional output patterns. Of course they also exhibit all the common features of structurally stable dynamical systems such as smooth changes under parameter variation. The structural stability makes it possible to fuse in input without destroying the autonomous dynamics of the system, i.e., the resulting dynamics is a combination of internal and external dynamics.

An interesting example of the use of oscillators in engineering is in the field of locomotion control in robots. Locomotion control is still a difficult and unsolved problem for robots with multiple degrees of freedom (e.g., legged robots). Locomotion requires multidimensional

coordinated periodic patterns that need to satisfy multiple constraints in terms of efficient locomotion, energy, and adaptation to complex terrain.

One approach to solving this problem relies on accurate models of the robot and environment dynamics to develop control laws for locomotion. These model-based methods have, however, significant difficulties dealing with environments that are hard to model properly (e.g., with complex terrains). An alternative approach is to use systems of coupled oscillators and to take inspiration from animal motor control. In vertebrate animals, an essential building block of the locomotion controller is the *central pattern generator* (CPG) located in the spinal cord. A CPG is a neural circuit capable of producing coordinated patterns of rhythmic activity in open loop, i.e., without any rhythmic inputs from sensory feedback or from higher control centers Delcomyn (1980); Grillner (1985). A CPG can be modeled as a system of coupled oscillators Kopell and Ermentrout (1988); Kopell (1988). The motivation for using CPG models in robotics is to produce the periodic patterns necessary for locomotion as limit cycles. If this is the case, the oscillatory patterns are robust against transient perturbations (i.e., they asymptotically return to the limit cycle), and this makes them well suited to deal with unexpected perturbations from the environment. Furthermore, the limit cycle can usually be modulated by some parameters, which offers the possibility to smoothly modulate the gait (e.g., increase frequency and/or amplitude) or even to induce gait transitions (i.e., bifurcations between different types of limit cycles; see for instance Collins and Richmond 1994). Finally, CPGs can readily integrate sensory feedback signals in the differential equations and show interesting properties such as entrainment by the mechanical body Taga (1998). Because of these interesting properties, CPGs are being increasingly used in robotics (see for instance Kimura et al. 1999; Wilbur et al. 2002; Endo et al. 2005; Buchli et al. 2006; Righetti and Ijspeert 2006b).

The paper is organized as follows. First, in Sect. 2 we introduce oscillators (i.e., limit cycle systems) using definitions and present a description of the very basic features common to *all* limit cycle systems. We then discuss the typical stability properties of oscillators, and this leads us to the formulation of two distinct coordinate systems, the phase–radius coordinate system (PRCS) and the Q coordinate system (the “physical” coordinate system), and their relationship. Examples of the abstract concepts using well-known oscillators will be given along the way. This discussion helps us to get clear what properties of an oscillator can be designed. Based on this discussion we then address the issue of how we can

construct an oscillatory system to exhibit specific properties. This part addresses the three core questions: What can we do (Sect. 3)? How can we do it (Sect. 4)? What are the resulting systems (Sect. 5)? Finally, we conclude with a general discussion of design choices and give an outlook on future research in this direction.

We assume some familiarity with basic concepts of nonlinear dynamical systems. A good introduction can be found in, e.g., Strogatz (1994); especially the concept of stability will be used extensively. The mathematical facts presented in this article are often not new (many are on a textbook level); however, the presentation of oscillators is new due to the focus on the generic properties of all oscillators. This leads to a novel discussion of oscillators centered around the transformation between a canonical system in the PRCS, in which it is particularly simple to discuss the influence of perturbation of oscillators, and the “physical” system, i.e., the traditional way of representing oscillators. To support the discussion, the new concept of radius isochrones is introduced. Finally, we would like to make a statement about mathematical detail and completeness. For the sake of clarity we do not discuss every subtlety and every case since we think that would dilute the clarity of the concepts. An excellent and very comprehensive introduction to oscillators and synchronization phenomena is Pikovsky et al. (2001).

2 Limit cycle systems

In this section we will introduce the *mathematical* concept of an oscillator. As will become clear, there are some subtle differences between what is commonly called an oscillator and the mathematical concept of an oscillator.

The presented concepts are key to all the design issues discussed later, such as the choice of type of oscillator, coupling, etc. This means that an engineer wanting to use an oscillator needs a thorough understanding of those concepts. This will allow him/her to choose the right type of oscillator and will give him/her the tools to engineer its properties.

Definitions To support our mathematical discussion of oscillators, we start with its definition.

Definition 1 *An oscillator is an autonomous dynamical system, i.e., a system of differential equations, with at least one limit cycle attractor. In other words, the solution of the system (after a transient time) is a closed cycle, which is asymptotically stable, i.e., if the system gets perturbed out of the limit cycle, then it returns back to it.*

Table 1 Nomenclature, conventions, and common abbreviations used to discuss oscillators in this article

$\mathbf{F}(\mathbf{q}, \rho)$	System of equations describing the dynamics of the system $\mathbf{F}(\mathbf{q}) = [f_1, \dots, f_D]^T$
\mathbf{q}	State vector $\mathbf{q} = [q_1, \dots, q_D]^T$
ρ	Vector of parameters
$\mathbf{p}(t)$	Vector of perturbations
D	System dimension
ϕ	Phase of oscillator
ω	Intrinsic frequency of oscillator
\mathbf{r}	Radius of oscillator $\mathbf{r} = [r_1, \dots, r_{D-1}]^T$
T	Period of oscillator
Ω_i	Instantaneous frequency
$\mathbf{e}_\phi, \mathbf{e}_r$	Unit vectors in direction of phase and radius
\mathbb{T}	Transformation from \mathbf{q} into $[\phi, \mathbf{r}]$
\mathbf{q}_∞	Set of points describing limit cycle
PRCS	Phase–radius coordinate system
QCS	\mathbf{q} -coordinate system
PS	Phase sensitivity
PRC	Phase response curve

We see that the limit cycle attractor is the defining property of an oscillator; hence the name limit cycle system is used as a synonym for oscillator.

This means the system has a self-sustained oscillatory behavior to which it returns after a transient perturbation. Oscillators thus possess an *intrinsic* period (and hence frequency) with which the system repeats the pattern of activity.

Thus, a linear second (or higher) order system cannot be an oscillator in that sense. It can only exhibit sustained oscillations with an oscillatory input in case the system is damped. If it is not damped, the system is just on the border between stability and instability and oscillations are not structurally stable (i.e., after a transient perturbation possibly another pattern is assumed). On the other side of the stability border the oscillations will increase to infinite amplitude.

Let us detail this definition a bit more. For this we need some nomenclature, which we list in Table 1.

With the help of those variables we can restate the above definition in more concise terms:

Definition 2 (equivalent to Def. 1) *If the dynamical system*

$$\dot{\mathbf{q}} = \mathbf{F}(\mathbf{q}) \quad (1)$$

has a stationary solution that is a closed curve and the solution is structurally stable, then (1) is an oscillator.

The limit cycle The set on which \mathbf{q} evolves is called the limit cycle, which we denote with \mathbf{q}_∞ . It is an attractor of dimension 1 (i.e., a curve), which is closed in itself, so it needs to be embedded in a space of dimension $D \geq 2$ (cf. Fig. 1). The fact that the attractor forms a

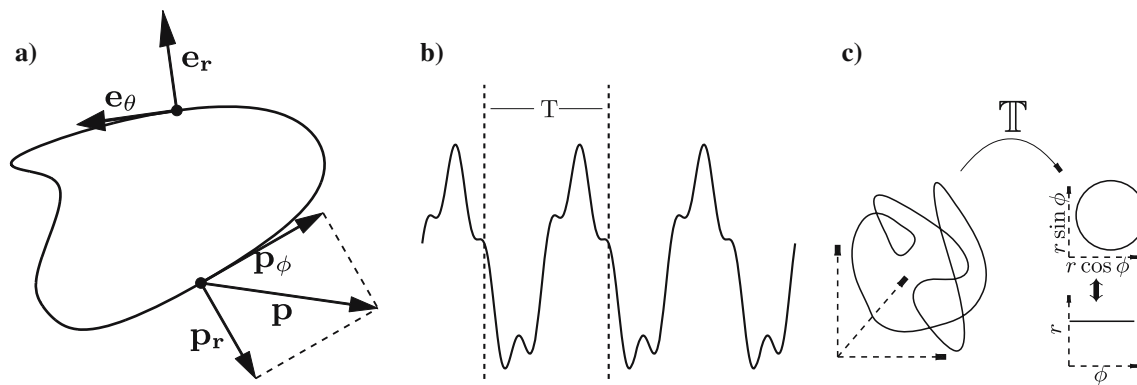


Fig. 1 **a** Schematic illustration of a limit cycle. It is a closed curve in phase space. The stability directions \mathbf{e}_ϕ and \mathbf{e}_r are illustrated as well as the projections \mathbf{p}_r and \mathbf{p}_ϕ of a perturbation \mathbf{p} , which has a direction in the phase space, onto those stability directions. **b** Time series of a hypothetical oscillator. There is a characteristic period T after which the activity of the oscillator, and with this the

time series, repeats. **c** The limit cycle is a 1-D manifold embedded in a D -dimensional space ($D \geq 2$); we can transform the system into a coordinate system in which the manifold shows particularly simple form and of which the stability directions constitute the base vectors

closed curve implies that the *time shift invariance* holds: if $\mathbf{q}(t)$ is a solution, then $\mathbf{q}(t) = \mathbf{q}(t + nT)$, where T is the time of repetition, i.e., the period of the system, and is inversely proportional to the intrinsic frequency ω of the system $T = \frac{2\pi}{\omega}$ (cf. Fig. 1b).

The fact of having a closed curve implies also a special stability property of the attractor. The flow described by the set of equations lets all solutions within the basin of attraction converge to the limit cycle. Perpendicular to the limit cycle the system is thus asymptotically stable. But the phase point moves along the limit cycle. In other words in every point on the limit cycle the flow is stable/contracting in to $D-1$ directions but drives the state in the one direction perpendicular to the other directions. This is the very essence of a limit cycle system; these stability properties are the only ones that allow for a closed 1-D attractor, and they are the key to understanding the properties of oscillators, their particular behavior, and phenomena such as synchronization.

We can thus distinguish two characteristic stability directions on the limit cycle by introducing a coordinate system of which one basis vector is tangential to the limit cycle, \mathbf{e}_ϕ , and $D-1$ vectors perpendicular to the limit cycle, which we do denote representatively by \mathbf{e}_r (cf. Fig. 1a).

As we will see in the next section, the marginally stable direction tangential to the limit cycle is of central importance in any discussion of oscillators.

The phase In every oscillator we can identify a variable (which does not necessarily correspond to a state variable but is a function of those) that grows *uniformly* in time and is interpreted *mod* 2π (or any other convention). This variable is called the phase of the oscillator.

The phase of the oscillator is a measure where the oscillator is in the cycle. We remember that the frequency of the oscillator is $\omega = \frac{2\pi}{T}$. We define the phase in the following way:

Definition 3 For the unperturbed system \mathbf{F} , the variable ϕ for which

$$\dot{\phi} = \omega (= \text{const}) \quad (2)$$

is called *phase of the oscillator* \mathbf{F} .

With the help of the phase, also the frequency can be defined:

Definition 4 The rate of change ω of ϕ in the unperturbed oscillator is the *intrinsic frequency of the oscillator*.

The reason why we define the frequency using the phase will become clear later when we discuss the oscillator under perturbations. It is important to note that ω is not always an explicit parameter. However, it is always a function of the parameters, $\omega = f(\rho)$.

Now, every oscillator can be transformed into a phase (ϕ)–radius (\mathbf{r}) coordinate system [Pikovsky et al. \(2001\)](#):

$$\dot{\phi} = \omega, \quad (3)$$

$$\dot{\mathbf{r}} = \mathbf{F}_r(\mathbf{r}), \quad (4)$$

where \mathbf{F}_r is the dynamical system describing the evolution of \mathbf{r} (Fig. 1c) and has a stable fixed point.

The vectors \mathbf{e}_ϕ and \mathbf{e}_r , introduced above based on stability considerations, form the basis vectors of the PRCS. In Eq. (3) the fact that the phase is a marginally stable variable ($\frac{d\phi}{dt} = 0$) is immediately evident. This coordinate system is the natural one to use in a discussion of

influences on the oscillator since the stability directions are decoupled.

At this stage it is also worth noting that in an unperturbed oscillator the phase *completely* describes the state of the system in the stationary regime (cf. phase oscillators in Sect. 3.1). This means we can write $\mathbf{r}_\infty = f(\phi)$.

It is important to realize that the phase is not necessarily proportional with time. This is only the case when the oscillator is unperturbed, where indeed ϕ is proportional to $t \bmod 2\pi$; but more importantly—and this is the key to the entrainment effects as we will see later—in the case of perturbations the phase and time “get decoupled,” i.e., the phase can be shifted forward or backward. The oscillations can be accelerated or decelerated. Designing entrainment and other aspects of the oscillator is all about designing these acceleration and deceleration effects. We will discuss this in more detail when we look at limit cycles under perturbations in Sect. 2.2. But first we need to complete our understanding of the PRCS.

The geometry of the limit cycle Normally the oscillator is not readily represented in the ideal PRCS (as in Eqs. (3)–(4)). We need to discuss the relationship between the oscillators’ representation in \mathbf{q} and in $[\phi, \mathbf{r}]$. As we will see, that is the key to understanding the behavior of the oscillator under perturbations.

If we transform the coordinate system and the metrics, we could possibly gain a simpler oscillator, but the complexity gets transferred to the coupling of the oscillator, i.e., the input and output functions. We will give an example when we discuss phase oscillators in Sect. 3.1.

But then how is the coordinate system usually determined? Let us reflect on the role of the state variables. Usually the state variables are defined by a physical interpretation or they have a concrete conceptual meaning such as a voltage or a chemical concentration, for instance, and that is the way the coordinate system gets defined. This coordinate system is the natural one to use to formulate the physical laws and interaction between the different physical entities (while, as we have seen, phase radius is the natural one to use to discuss perturbation because of the separation of variables according to stability properties). And in modeling it is usually the case that the inputs and outputs of an oscillator are formulated in the physical coordinate system.

But, for engineering, if we want to exploit some of the abstract properties of the oscillator, we are not bound to an interpretation of the variables. Thus it can be useful to formulate the inputs and outputs in the PRCS or any other suitable coordinate system.

We name thus two coordinate systems, the phase-radius coordinate system (PRCS) and the coordinate system of \mathbf{q} (QCS).

The transformation from QCS into PRCS is given by \mathbb{T} :

$$[\phi, \mathbf{r}]^T = \mathbb{T}(\mathbf{q}). \quad (5)$$

Let us consider in more detail this transformation and some of its properties. The transformation can be split up into components, namely, the transformation from \mathbf{q} into ϕ : $\phi = T_\phi(\mathbf{q})$ and the transformation from \mathbf{q} into $\mathbf{r} = T_r(\mathbf{q})$, thus

$$\mathbb{T} = \begin{bmatrix} T_\phi \\ T_r \end{bmatrix}.$$

Since we are interested not only in the transformation of the state variables but also in the transformation of the dynamics under this transform, let us investigate the derivatives of the transformed coordinates. We do this using ϕ as an example since it works the same way for all state variables:

$$\phi = T_\phi(\mathbf{q}), \quad (6)$$

$$\begin{aligned} \Rightarrow \dot{\phi} &= \frac{dT_\phi(\mathbf{q})}{dt} = \frac{\partial T_\phi(\mathbf{q})}{\partial \mathbf{q}} \dot{\mathbf{q}} \\ &= \frac{\partial T_\phi(\mathbf{q})}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}). \end{aligned} \quad (7)$$

We see that T_ϕ is intrinsically defined (by the fact that Eq. (7) must be equivalent to ω , which is a constant), but there is some freedom in the choice of T_r . To remove this ambiguity, we define the behavior of an oscillator in the canonical PRCS as

$$\dot{\phi} = \omega, \quad (8)$$

$$\dot{\mathbf{r}} = 1 - \mathbf{r}. \quad (9)$$

This is somewhat arbitrary, but the choice will become clear later in the discussion of the relationship between PRCS and QCS. At this place its choice is already partially motivated by Eq. (9) representing the simplest dynamical system with stable, nonzero fixed-point behavior, where the fixed point is $\mathbf{r} = 1$.

The inverse of \mathbb{T} , \mathbb{T}^{-1} , transforms the system from the PRCS into the QCS. This means that, by designing \mathbb{T}^{-1} , the PRCS can be mapped into any type of oscillator.

Example Consider the transformation of r into r' , given by $r' = (r - 1)^g + r_0$. This transforms the canonical oscillator (Eqs. (8) and (9)) into

$$[\dot{\phi}, \dot{r}']^T = [\omega, -g(r' - r_0)]^T. \quad (10)$$

We can now transform this system further by applying the well-known transformation from a Polar coordinate system into the Cartesian coordinate system: $q_1 =$

$r' \cos \phi$, $q_2 = r' \sin \phi$. By this transform we yield the following system:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} g \left(\frac{r_0}{\sqrt{q_1^2 + q_2^2}} - 1 \right) q_1 - q_2 \omega \\ g \left(\frac{r_0}{\sqrt{q_1^2 + q_2^2}} - 1 \right) q_2 + q_1 \omega \end{bmatrix}. \quad (11)$$

We have thus transformed the canonical oscillator into a phase oscillator in a Cartesian coordinate system. The radius and convergence rate of the oscillator can be controlled by r_0 and g , respectively.

While here we can express the transform \mathbb{T}^{-1} and its inverse by rather simple mathematical expressions, this is usually not possible. Furthermore, the transform might often not be expressible in a closed analytical form.

2.1 Graphical assessment of \mathbb{T}

We will now discuss a graphical way of assessing the relationship between PRCS and QCS, i.e., \mathbb{T} , for a given oscillator. For this the notion of time becomes important, i.e., at what velocity the phase point moves through the phase space. What specifies the equivalence of a point in QCS and a point in PRCS? If for time t_0 $\mathbf{q}(t_0) = \mathbb{T}^{-1}(\phi(t_0), \mathbf{r}(t_0))$, then for all time $\mathbf{q}(t) = \mathbb{T}^{-1}(\phi(t), \mathbf{r}(t))$. This means if we could find a way of comparing the development of the two points in both coordinate systems at regular intervals and repeat this for different initial conditions, then we would get an idea of \mathbb{T} .

The activity of the oscillator can be plotted in the QCS (phase portrait) in which the limit cycle will show up as a closed curve. But the information about the phase velocity is lost. So even though we know that, for example, the limit cycle in the QCS corresponds to the limit cycle in the PRCS, we do not know which point on the limit cycle in one coordinate system corresponds to which point in the other one. While in PRCS the phase moves along the limit cycle with constant velocity ω , in general, for an arbitrary oscillator \mathbf{F} in the QCS, the phase point will not move along the limit cycle in the phase space with a constant velocity. Thus, the phase does not correspond to the simple “position” on this curve (i.e., an infinitesimal part of the curve does not correspond to the same infinitesimal part of ϕ , $d\mathbf{q} \neq d\phi$). In order to investigate this relationship, we could plot various points that have a common $\Delta\phi$.

However, the phase is only uniquely defined on the limit cycle, but we would like to get a global idea of the transformation. We can generalize the notion of a phase outside the limit cycle by the concept of *isochrones*. We follow the definition of [Pikovsky et al. \(2001\)](#) and

generalize the phase with the help of the cycle time T , i.e., the mapping $\mathbf{q}(t) \rightarrow \mathbf{q}(t + T)$.

Definition 5 *The set of points that are invariant under the mapping $\mathbf{q}(t) \rightarrow \mathbf{q}(t + T)$ and cross the limit cycle at \mathbf{q}^* (i.e., \mathbf{q}^* is a fixed point of the mapping) are called an isochrone through \mathbf{q}^* .*

In other words, all points of the phase space that converge to have the same phase on the limit cycle form an isochrone.

If we plot isochrones for every $\Delta\phi = \text{const}$, they show the relationship of the phase with the geometry of the system in the QCS (cf. Fig. 2). Where they are tightly spaced the phase point moves slowly; thus values of different phases are more tightly spaced. If they are equally spaced on the limit cycle, the lengths of an arc of the limit cycle is proportional to $\Delta\phi$ ($d\phi = \frac{1}{S} 2\pi ds$, where s is the arc length and S the total length of the limit cycle). If the isochrones are straight (as in Fig. 2b), this means that the differential equations for ϕ and \mathbf{r} are decoupled, and the transformation \mathbb{T} corresponds to a transformation from Cartesian into polar coordinates.

Now, the isochrones give us an idea as to how ϕ is embedded in QCS. But to complete our picture of the transformation \mathbb{T} , we need to get an idea of how \mathbf{r} is embedded in this coordinate system.

For this we define the radius isochrones, which will give us an idea as to how \mathbf{r} evolves over time.

Definition 6 *The set of points $\mathbf{q}(t)$ satisfying $\text{dist}(\mathbf{q}(t + \Delta t), \mathbf{q}_\infty) = \epsilon$ are called a radius isochrone with Δt .*

Function *dist* denotes the distance between point \mathbf{q} and the limit cycle. Intuitively, this means all points that converge to the limit cycle at the same time form a radius isochrone. We see that the definition of the radius isochrone implies a distance measure from the limit cycle. This distance can be defined in different ways. We use the perpendicular direction to the limit cycle and consider it to be converged to the limit cycle when it enters the “tube” of radius $\epsilon \ll 1$ around the limit cycle.

We can use the isochrones and the radius isochrones to get an idea as to how the abstract phase–radius oscillator is embedded in the coordinate system for \mathbf{q} . The time to get from one isochrone to the next is constant, and the time from one radius isochrone to the next is also constant. Radius isochrones give information about the rate of convergence; tightly space radius isochrones mean a slow convergence to the limit cycle, while widely spaced ones mean fast convergence (see Fig. 2 and its description).

If we plot isochrones equally spaced in ϕ and radius isochrones for equally spaced Δt for the oscillator in the canonical PRCS (Eqs. (8) and (9)), we get a rectangular

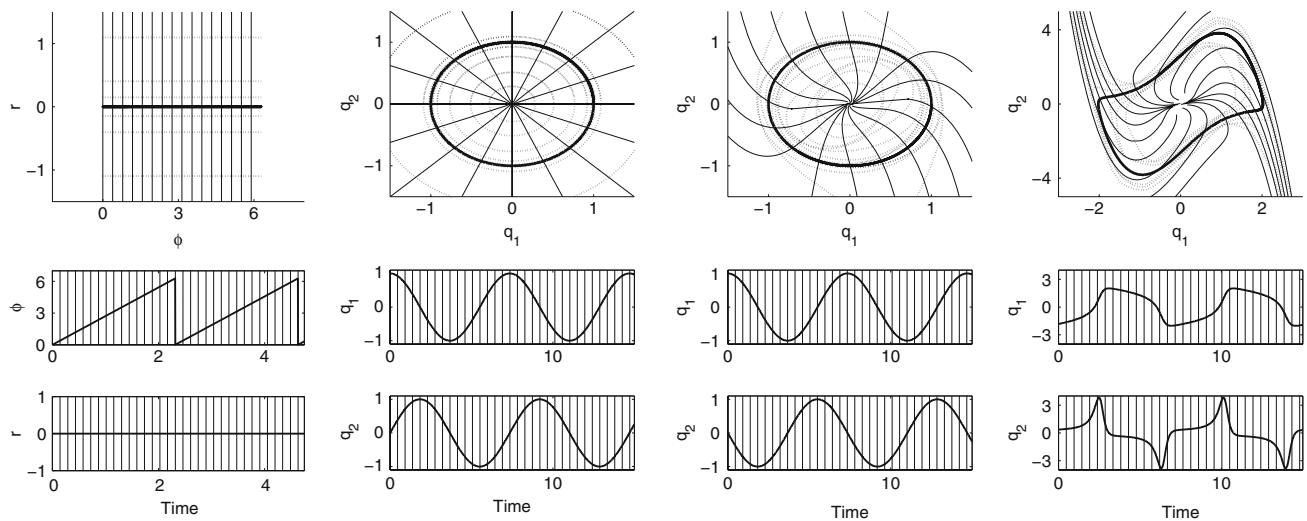


Fig. 2 Illustrating the transformation \mathbb{T} with the help of isochrones and radius isochrones. For each oscillator 16 equally spaced isochrones are used and a varying number of radius isochrones with a given Δt are plotted in the phase portrait (*upper panels*) and below the time series are shown. The *fine vertical lines* indicate the isochrones (indicating only the temporal position on the time series). We see that, while they are always equally spaced in the time series plot, in the phase plot this is not necessarily the case. **a** Phase plot of canonical oscillator in PRCS (Eqs. (3)–(4)). The isochrones form straight and equidistant vertical lines. The radius isochrones ($\Delta t = [5, 6, 7]s$) form exponentially spaced straight horizontal lines. **b** Hopf oscillator (Eqs. (30) and (31)). The isochrones form straight rays at equal angles, which reflects the polar interpretation of ϕ in the transformation. The radius isochrones ($\Delta t = [0.7, 1.4, 2.1]s$ outside of the limit cycle and

$\Delta t = [0.7, 1.4, 2.1, 3.5]s$ inside) reflect the third-order convergence behavior of the radius. The time series reflects the harmonic nature of the limit cycle. **c** Energy oscillator (Eqs. (22) and (23)). Due to the appearance of the nonlinear energy term, only in the first ODE does the system lose its circular symmetry. The isochrones and the radius isochrones ($\Delta t = [2, 2.4, 2.8, 3.2]s$ outside of the limit cycle and $\Delta t = [2, 2.4, \dots, 5.2]s$ inside) get a characteristic deformation. **d** van der Pol oscillator (Eqs. (42) and (43)). This is a strongly nonlinear oscillator. This fact is reflected in the strong deformation of the isochrones. The strong deformation of the radius isochrones $\Delta t = [0.3, 0.6, 0.9]s$ away from the limit cycle in the *upper left* and *lower right corner* of the figure indicates the rapid convergence of the system in that region. It is immediately apparent that the transformation from this QCS into the PRCS is a very complicated one

grid as seen in Fig. 2a with equally spaced vertical lines and exponentially spaced horizontal lines. This means that whenever we see those characteristics in an oscillator, we know it behaves like a canonical oscillator (i.e., exponential convergence toward the limit cycle and isochrone behavior).

2.2 Limit cycles under perturbations

Considering the fact that we can transform *every* oscillator into a PRCS, it gets clear that we can consider all *unperturbed* oscillators to be equivalent up to a transformation \mathbb{T} . They only differ in the way in which a (physical) input to the system affects its dynamics, i.e., how the input output space is related to the PRCS.

So far we have considered the autonomous, i.e., unperturbed, oscillator. However, the advantage of the use of oscillators (e.g., vs. function-based approaches) becomes effective only when coupling them to perturbations. This can mean the oscillator is coupled to some external input and output or to other oscillators.

If the input and outputs are formulated in a QCS, then, with respect to their influence on the limit cycle, they undergo the same transformation \mathbb{T} . We thus must investigate limit cycle systems under perturbations. Therefore, we must consider the single oscillator as a nonautonomous system:

$$\dot{\mathbf{q}} = \mathbf{F}(\mathbf{q}, \mathbf{p}(t)). \quad (12)$$

It is important to stress that a perturbation can theoretically have any arbitrary functional form, and a perturbation is also not limited to acting on the first state variable only. However, a discussion of the different types of perturbations is beyond the scope of this article. We will focus on an often used form of perturbation, the additive perturbation:

$$\dot{\mathbf{q}} = \mathbf{F}(\mathbf{q}) + \mathbf{p}(t). \quad (13)$$

We will, however, realize that the additive form in a QCS transforms into a more complicated functional form in a PRCS. The oscillator in the PRCS becomes

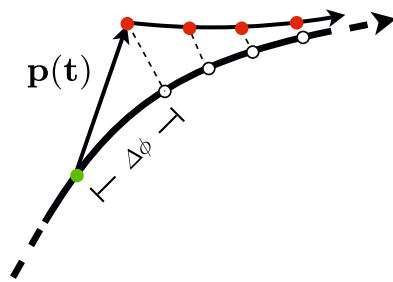


Fig. 3 Effect of a small pulselike perturbation on the limit cycle. The perturbation $\mathbf{p}(t)$ arrives when the phase point is at the position marked by the *green dot*. The phase point is then pushed back to the limit cycle by the stability properties of the system, i.e., it approaches the limit cycle asymptotically. However, it retains a phase difference ($\Delta\phi$) in comparison with the unperturbed reference system. The phase difference can be of different amplitude and sign depending on the direction of the perturbation *and* the state the system currently is in when the perturbation arrives. Understanding this fact is key to understanding synchronization phenomena

$$\dot{\phi} = \omega + p_{\phi}, \quad (14)$$

$$\dot{\mathbf{r}} = \mathbf{F}_r(\mathbf{r}, \phi) + \mathbf{p}_r, \quad (15)$$

where p_{ϕ} is the component of the perturbation acting on the phase and \mathbf{p}_r is the component of the perturbation acting in the direction of the radius.

Expressed with the help of transformation \mathbb{T} , we yield for the additive case (i.e., Eq. (13))

$$\Rightarrow \dot{\phi} = \frac{\partial T_{\phi}(\mathbf{q})}{\partial \mathbf{q}} \mathbf{F}(\mathbf{q}) + \frac{\partial T_{\phi}(\mathbf{q})}{\partial \mathbf{q}} \mathbf{p}. \quad (16)$$

On the limit cycle $\mathbf{e}_{\phi} = \mathbb{T}(\mathbf{q})$, the relationship between the PRCS and the QCS is thus the determinant for the effect of the perturbations on the oscillator.

Let us give a geometric intuition, which lets us easily derive p_{ϕ} , without relying on transformation \mathbb{T} . But this derivation is only valid on the limit cycle, while the above transforms are more general.

To arrive at this simpler derivation, it is important to realize that every perturbation has a direction in the phase space.

Consider a pulselike additive input, i.e., an infinitely short input at time t_p . The perturbation will bring the phase point away from the limit cycle. The stability properties will bring it back to the limit cycle, but at another position, relative to the unperturbed system. Thus the phase of the system $\phi(t_{p+})$ is not the same as before the perturbation $\phi(t_{p-})$; the phase is reset, hence the term “phase resetting” (cf. Fig. 3).

Thus, for small perturbations the effect that remains in the system is the effect of the perturbation in the direction of the phase \mathbf{e}_{ϕ} , which is the direction tangential to the limit cycle or, equivalently, the direction $\dot{\mathbf{q}}$:

$$\mathbf{e}_{\phi} = \frac{\dot{\mathbf{q}}}{|\dot{\mathbf{q}}|}. \quad (17)$$

Therefore, the effective perturbation on the phase is

$$p_{\phi} = \mathbf{p} \cdot \mathbf{e}_{\phi}. \quad (18)$$

The derivative of the phase becomes

$$\dot{\phi} = \omega_0 + \mathbf{p} \cdot \mathbf{e}_{\phi}. \quad (19)$$

Thus we found the sensitivity of the phase to perturbations:

$$S_p(\mathbf{p}) \doteq \frac{p_{\phi}}{|\mathbf{p}|} = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \mathbf{e}_{\phi} = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \frac{\dot{\mathbf{q}}}{|\dot{\mathbf{q}}|}. \quad (20)$$

This means that, depending on the state of the oscillator, the same perturbation can have different influences; at one stage it can speed up the oscillator slightly, while at another state it can slow it down. If the sum of this acceleration or deceleration is nonzero, then this leads to entrainment effects.

The sensitivity of the phase to perturbations is summarized in the phase-reset curve (PRC) and its generalization, the phase sensitivity (PS). The PRC is a function that describes the effect of a unitary pulselike perturbation as a function of the phase of the oscillator it arrives at. In other words it tells how much the phase is shifted by that perturbation.

The PS generalizes this idea as it is not restricted to a single pulselike perturbation per cycle but is an “instantaneous” description of the effect of perturbations. Due to its importance in the discussion about the influence of perturbation on oscillators, much research has been done mainly on PRC but also on PS (see [Pikovsky et al. \(2001\)](#) and references therein). For an example of the derivation phase locking with the presented tools, see [Buchli and Ijspeert \(2004\)](#).

It can be difficult or impossible to obtain the analytical form of the PRC or the PS. However, with the directional idea introduced in Eq. (20) it can be estimated from numerical integration. It can also be measured to a certain extent in real-world systems. We will come back to the role of the PRC/PS when discussing the design of entrainment effects in Sect. 5.

Since the frequency of the oscillator corresponds to the rate of change of the phase, we see that in a perturbed oscillator the observed frequency is not necessarily the same anymore.

Definition 7 The instantaneous frequency Ω_i is defined as the momentary rate of change of the phase

$$\Omega_i = \dot{\phi}. \quad (21)$$

While for the autonomous oscillator the instantaneous frequency is equal to the intrinsic frequency and constant ($\Omega_i = \omega = \text{const}$), in the perturbed oscillator the instantaneous frequency is not equal to the intrinsic frequency ($\Omega_i \neq \omega$) and is also a function of time ($\Omega_i = f(t)$). Ω_i is the frequency that is observed or measured at any given time (e.g., by such methods as windowed FFT, spectrograms, or wavelets). ω is the parameter, while Ω_i is a variable that can be decoupled from ω by a perturbation.

Remembering the relation $\Omega_i = \dot{\phi}$, this also means that the observed frequency is *not the same as the intrinsic frequency*: if we have phase locking (i.e., when the difference between the phase of the oscillator and the phase of the perturbation remains bounded), the observed frequency will be the frequency of the perturbation. Consider as an example two mutually connected oscillators with intrinsic frequencies $\omega_{1,2}$ that are different but close enough to have mutual entrainment. The observed frequency Ω_i will be the same for both oscillators but different from the two intrinsic frequencies, i.e., $\Omega_i \neq \omega_{1,2}$ (it will be between the two).

Thus, it can be said that the oscillator gets entrained by the perturbation. It could also be said that it adapts, but this change is only temporary, i.e., reactive. If the input is switched off, then the system immediately returns to its intrinsic dynamics, and there is no memory of the input and no lasting change to the dynamics.

Further, there can be an influence on the radius by the perturbation that can also be exploited. The stability directions here are, however, less special, so this usually reduces to quite standard treatment of ODEs with fixed points under perturbation. Note that the behavior of the radius under perturbation can still be difficult, especially when there is a strong deformation of the radius isochrones in the QCS (e.g., as for the van der Pol oscillator, cf. Fig. 2). Such a deformation means that the convergence behavior is very nonuniform and a perturbation has a completely different effect on \mathbf{r} depending on where it arrives.

Thus, summarizing the findings of this section, it must be realized that for the design of an oscillator it is important to know the stability properties and the effect of perturbations in the coordinate systems of the stability directions. Simply put, if we want to change the radius only, then the perturbation needs to act perpendicular to the limit cycle, i.e., move the point on the same isochrone. If we want to affect the phase only, then we

need to move the phase point tangential to the limit cycle, move it on the radius isochrones.

3 The design space

We realize that the oscillator can be completely reduced to the PRCS and the inputs can be formulated in that system. We can then possibly use the transformation \mathbb{T}^{-1} into a given interpretation coordinate system to talk about the oscillator in a physically or conceptually more meaningful coordinate system. Thus all design choices will in one way or another investigate the effects of perturbation in the PRCS (and then possibly backwards via \mathbb{T} , and what this means for the behavior of the oscillator in the QCS).

We are now at a stage where we can discuss *what can be designed in an oscillator*. There are three basic ways to define an oscillator and its input and output.

1. Choose an ODE system $\mathbf{F}(\mathbf{q}, \rho)$ and functional form of input and output, i.e., $\mathbf{F}(\mathbf{q}, \rho) \rightarrow \mathbf{F}(\mathbf{q}, \rho, \mathbf{p})$. Then work out the relation of \mathbf{F} to $[\dot{\phi}, \dot{\mathbf{r}}]$, i.e., we work out \mathbb{T} (or parts of it); the transformation is implicitly specified.

Here we need to convince ourselves either analytically or at least numerically that $\mathbf{F}(\mathbf{q}, \rho)$ represents indeed a structurally stable oscillator. Then we are sure about the existence of the PRCS and the stability properties that have been discussed.

Example:

$$\dot{q}_1 = -\alpha \frac{q_1^2 + q_2^2 - E}{E} q_1 - q_2 + p_x(t), \quad (22)$$

$$\dot{q}_2 = q_1. \quad (23)$$

This oscillator's steady-state solution is $\mathbf{q}_\infty = [\sqrt{E} \cos(t + t_0), \sqrt{E} \sin(t + t_0)]$. Structural stability is most easily shown by transforming the coordinates into polar coordinates and showing that the ODE for the radius has a stable fixed point. We can then show with rather straightforward analysis that in a stationary regime

$$\dot{\phi} = \omega + \sin(\phi) p_x(t). \quad (24)$$

2. Use an ideal phase or phase-radius oscillator; given input and output directly in this coordinate system, the transformation is thus implicitly specified by this choice.

Note that the transformation does not necessarily have to be fully specified, e.g., if only a scalar output

is needed, it suffices to define a part of the transformation.

Example: Choose a phase oscillator and add a non-linearity in the input $\dot{\phi} = \omega + \sin(\phi)p(t)$ that will synchronize the phase oscillator to $p(t)$ if frequencies are close. The output is chosen to be $o = \sin(\phi)$.

3. *Specify input in QCS and specify \mathbb{T}^{-1} explicitly.*

Example: Choose $\mathbf{q} = [q_1, q_2]$, a Cartesian coordinate system, and \mathbb{T} as the transformation into a PRCS, i.e., $T_\phi = \arctan \frac{q_1}{q_2}$ (arctan denotes the four-quadrant arcus-tangent), $T_r = \sqrt{q_1^2 + q_2^2}$. The input acts on the first state variable only: $\mathbf{p} = [p_1, 0]$. We can split up the relevant term from Eq. (16) in the following way:

$$\frac{\partial T_\phi(\mathbf{q})}{\partial \mathbf{q}} = [T_{\partial q_1}, T_{\partial q_2}]^T. \quad (25)$$

Since $\mathbf{p} = [p_{q_1}, 0]^T$, we can write

$$p_\phi = [T_{\partial q_1}, T_{\partial q_2}] \cdot [p_{q_1}, 0]^T = T_{\partial q_1} p_{q_1}. \quad (26)$$

Using $\phi = \arctan \frac{q_1}{q_2}$

$$\begin{aligned} T_{\partial q_1} &= \frac{d}{dq_1} \arctan \frac{q_1}{q_2} = \frac{1}{1 + (\frac{q_2}{q_1})^2} \frac{-q_2}{q_1^2} \\ &= -\frac{r \sin \phi}{r} = \sin \phi \\ \Rightarrow p_\phi &= -\sin \phi p_{q_1}. \end{aligned}$$

As one can see, the three examples are equivalent in their phase behavior. We also see that it is often not needed to have complete knowledge of the theoretical transformation \mathbb{T} .

The use of method 1 is widespread in the literature, where often some complicated \mathbf{F} out of the modeling literature is used. Immediately we realize that for engineering this is often not the best choice.

By introducing the stability directions and the geometrical aspects of interpretation of the PRCS (i.e., \mathbb{T}), we can identify two somewhat orthogonal design axes: (1) timing: a design that influences the phase of the oscillator, here we have (a) the influence on the (relative) phase, (b) instantaneous frequency, and (c) average frequency. (2) A design that influences the geometry of the oscillator. Here we have (a) influences on r directly and (b) the design of \mathbb{T}^{-1} , i.e., the output filter. This is a very interesting and important result, since orthogonal design axes are extremely helpful for engineering tasks. This means we can decouple the influence of parameters

on the outcome. In other words, by choosing a more suitable coordinate system, those influences get decoupled while in the original coordinate system they are not.

To detail this further, in what follows we address some common design goals and discuss how they translate into properties that the system has to exhibit (summary in Table 2).

1. *Specification:* Unperturbed average frequency: $\bar{\Omega}_i = \omega$. Average observed frequency should be equal to the intrinsic frequency ω .

Required property: $\int_0^{2\pi} p_\phi = 0$

We need the effect on the phase to have on average zero effect.

2. *Specification:* Frequency locking with an external signal of frequency ω_F , i.e., $\bar{\Omega}_i = \omega_F$.

Required property: $\frac{1}{2\pi} \int_0^{2\pi} p_\phi = \omega_F - \omega (= \omega_d)$ if

$\bar{\Omega}_i = \omega_F$. In differential terms: show that $\dot{\phi}_d = \dot{\phi} - \dot{\phi}_F$ is bounded $\int_0^{2\pi} \dot{\phi}_d = \int_0^{2\pi} p_\phi + \omega_d = 0$; the perturbation in the phase needs on average to make up for the differences between the intrinsic frequency and the frequency of the perturbation.

3. *Specification:* Phase locking with arbitrary phase lag: $\phi_d = \Phi_r$.

Required property: $\dot{\phi}_d(\Phi_r) = p_\phi(\Phi_r) + \omega_d = 0$ and $\frac{d}{d\phi} \dot{\phi}_d|_{\Phi_r} < 0$. The differential equation for ϕ_d needs a stable fixed point at Φ_r . This is achieved when the perturbation at every instant cancels for the differences between the intrinsic frequency and the frequency of the perturbation. Note that phase locking implies frequency locking.

4. *Specification:* Arbitrary instantaneous frequency: $\dot{\phi}(t) = \Omega_r(t)$.

Required property: $p_\phi = \Omega_r(t) - \omega$.

5. *Specification:* Arbitrary T -periodic output signal shape. *Required property:* An appropriate output function or dynamical system must be found, i.e., this leads to filter design, or the design of \mathbb{T}^{-1} .

6. *Specification:* Arbitrary form of the limit cycle in QCS, i.e., $\mathbf{q} = \mathbf{r}(t)$ —one or several state variables should follow a reference trajectory. This means we want a general form of the limit cycle: we have a closed curve in QCS that should be the limit cycle. *Required property:* We need thus to design the stability directions to be tangent, i.e., on the curve the flow must have the direction of the tangent. Normal to the curve the flow must be stable. If $D(\cdot)$ denotes the tangential and $\mathbf{n}(\cdot)$ the normal direction to a curve, we require that (i) the flow be tangential to the reference trajectory $\frac{\mathbf{F}(\mathbf{r}(t))}{|\mathbf{F}(\mathbf{r}(t))|} = \frac{D(\mathbf{r}(t))}{|D(\mathbf{r}(t))|}$ and (ii) contracting be perpendicular to the limit-cycle $\frac{\partial \mathbf{n}(\mathbf{r}(t))}{\partial \mathbf{n}} < 1$.

Table 2 Common design goals and the required oscillator properties

Specification	Required property
Unperturbed average frequency: $\bar{\Omega}_i = \omega$	$\int_0^{2\pi} p_\phi = 0$
Frequency locking: $\bar{\Omega}_i = \omega_F$	$\frac{1}{2\pi} \int_0^{2\pi} p_\phi = \omega_F - \omega (= -\omega_d)$ if $\Omega_i = \omega_F$, in differential terms: show that $\phi_d = \phi - \phi_F$ is bounded $\int_0^{2\pi} \dot{\phi}_d = \int_0^{2\pi} p_\phi + \omega_d = 0$
Phase locking: $\phi_d = \Phi_r$	$\dot{\phi}_d(\Phi_r) = p_\phi(\Phi_r) + \omega_d = 0$ and $\frac{d}{d\phi} \dot{\phi}_d(\Phi_r) < 0$ (phase locking implies frequency locking)
Spec. instantaneous frequency: $\dot{\phi}(t) = \Omega_r(t)$	$p_\phi = \Omega_r(t) - \omega$
Arbitrary output signal shape: $x(\phi) = x_r(\phi)$ or $x(\mathbf{q}) = x_r(\mathbf{q})$	Appropriate function, i.e., filter
Arbitrary form of the limit cycle in QCS: $\mathbf{q}_\infty = \mathbf{q}_{ref}(t)$	Stability directions, i.e., $\frac{\mathbf{F}(\mathbf{q}_{ref})}{ \mathbf{F}(\mathbf{q}_{ref}) } = \frac{D(\mathbf{q}_{ref})}{ D(\mathbf{q}_{ref}) }$ and $\frac{\partial \mathbf{n}(\mathbf{q}_{ref})}{\partial \mathbf{n}} < 1$

Remember that if we decide to work with a QCS, the perturbation on the phase is $p_\phi = f(\mathbb{T})$; thus the properties to be satisfied to meet the above design goals contain the relationship between a chosen coordinate system (QCS) and the PRCS.

The properties listed above will not necessarily allow a directed design without further assumptions and simplifications as they can lead to very difficult expressions. In the second part of the article we will address how some of the above properties have been designed in previous works.

3.1 Classes of limit cycles

Here we list a few important classes of oscillators, with properties in terms of the above discussed topics. They are loosely ordered from “simple” to more “complicated” (in the sense of \mathbb{T}). As we saw above, the classification only makes sense for a given coordinate system. We present them in the coordinate system in which they are usually used and additive input is used.

Phase oscillator Probably the simplest type of oscillator; where the radius is completely neglected, only the phase is retained. The phase oscillator is a linear system defined on the circle \mathbb{S}^1 instead of the Euclidean space \mathbb{R}^1 , which implies a closeness of the solution, and with that the system fulfills our definition of an oscillator.

Properties The phase oscillator is essentially a first-order linear differential equation; the nonlinearity lies in the interpretation of the phase modulo 2π and input/output relation. It is the most abstract oscillator. An important assumption for the applicability of the phase oscillator is that the limit cycle is strongly damped, i.e., the phase point is always on the limit cycle (or very close to it).

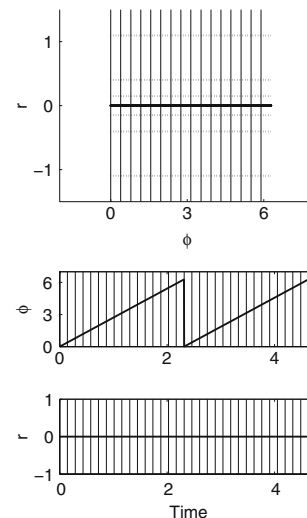
The output signal shape cannot change based on the input (other than direct functional coupling).

References: Phase oscillators were and are the “work-horse” to investigate synchronization effects, e.g., the Kuramoto oscillators Kuramoto (1984); Strogatz (2000).

Equations

$$\dot{\phi} = \omega + p(F(t), \phi) \quad (27)$$

Illustration



Isochronous oscillator

Properties Isochronous oscillators have straight isochrones, i.e., they are perpendicular to the limit cycle.

Equations

In the simplest case, the isochronous oscillator is a linear differential equation system. However, it is defined not on the Euclidean \mathbb{R}^D space but on $\mathbb{S}^1 \times \mathbb{R}^{D-1}$:

$$\dot{\phi} = \omega \mod 2\pi, \quad (28)$$

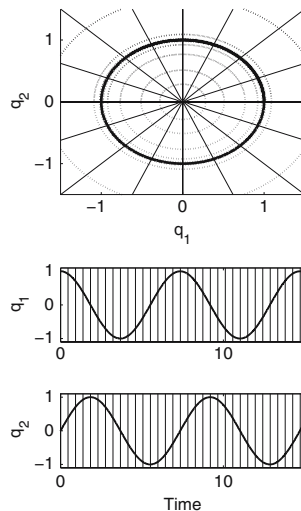
$$\dot{\mathbf{r}} = f(\mathbf{r}_0, \mathbf{r}, \phi), \quad (29)$$

where $f(\mathbf{r}_0, \mathbf{r})$ is a differential equation with stable fixed point \mathbf{r}_0 . The interpretation of ϕ modulo 2π , e.g., in Cartesian coordinates, makes it an oscillator, e.g., the Hopf oscillator:

$$\dot{x} = (\mu - r^2)x + \omega y, \quad (30)$$

$$\dot{y} = (\mu - r^2)y - \omega x. \quad (31)$$

Illustration



References The oscillator in Buchli and Ijspeert (2004) is isochronous.

Amplitude-controlled phase oscillator (ACPO)

The ACPO is the extension of the phase oscillator with a radius. The radius is controlled by a differential equation with a fixed-point attractor.

Properties The ACPO is simple in the sense that the phase shows up as an explicit state variable. This often allows for analytical treatment Buchli and Ijspeert (2004). However, to achieve higher-order locking, the input needs to generate opportunities for these locking regimes, in the sense that it needs to generate higher-order harmonics or subharmonics of the input. In contrast to the isochronous oscillator, the differential equation for the radius depends on the phase.

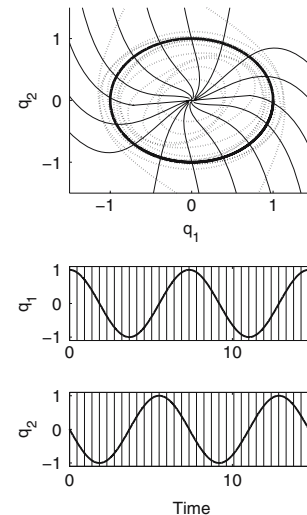
Equations

$$\dot{\phi} = \omega, \quad (32)$$

$$\dot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \phi), \quad (33)$$

where f is a nonlinear function with a stable fixed point.

Illustration



Harmonic oscillator An oscillator with a harmonic limit cycle, i.e., stationary solution $\mathbf{q}_\infty(t) = [r \cos(t), r \sin(t)]$. Note that this term conflicts with the common use of describing a linear second-order system. Such a system is, however, not an oscillator by our definition as discussed above.

Properties Due to its harmonic limit cycle some analytical results on the PS/PRC are possible and thus closed-form solutions for phase relationships and locking behavior can be derived.

Typical for harmonic oscillators is a possible description in Cartesian coordinates where the linear second-order oscillatory system shows up with an addition of nonlinear terms stabilizing the radius.

Equations

$$\dot{q}_1 = \omega q_2 + f_1(\mathbf{q}), \quad (34)$$

$$\dot{q}_2 = -\omega q_1 + f_2(\mathbf{q}). \quad (35)$$

See Eqs. (22) and (23) for a concrete example.

Illustration See illustrations for the isochronous oscillator and ACPO, which are both harmonic oscillators.

References The Hopf oscillator Hopf (1942) and the oscillators in Ijspeert et al. (2005); Buchli and Ijspeert (2004) are harmonic oscillators.

Piecewise linear system A system constructed of a set of linear systems of the same order, one of which is always active depending on the conditions of the state variables.

Properties A piecewise linear system is rather straightforward to design and analyze (piecewise

solution), though it can be problematic to simulate. The physical interpretation of the switching effect is a fast effect whose dynamics is lumped into the switch.

Equations Consider the oscillator from [Taga \(1994\)](#):

$$\dot{q}_1 = -q_1 - w \max(q_3, 0) - \beta q_2 + 1, \quad (36)$$

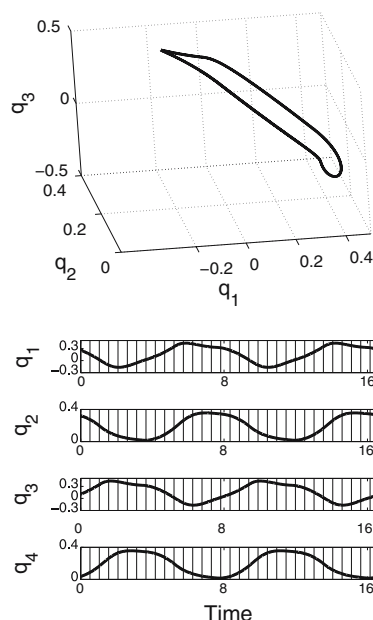
$$\dot{q}_2 = -q_2 + \max(q_1, 0), \quad (37)$$

$$\dot{q}_3 = -q_3 - w \max(q_1, 0) - \beta q_4 + 1, \quad (38)$$

$$\dot{q}_4 = -q_4 + \max(q_3, 0). \quad (39)$$

The system is switched whenever one of the state variables $q_{1,3}$ crosses 0, so the above form is a short form to describe four different systems and oscillates for certain parameter values (e.g., $w = 2.5, \beta = 2$).

Illustration



References The well-known Matsuoka oscillators [Matsuoka \(1985\)](#), applications of the Matsuoka oscillator in [Taga \(1994\)](#); [Fukuoka et al. \(2003\)](#). Such systems are also known as switched linear systems or hybrid systems.

Linear system with reset A linear, often second-order system that is reset if a variable passes a certain threshold. Such systems are an approximation of the relaxation oscillators where we can consider the fast effect to be infinitely fast.

Properties Between resets such systems behave like a linear system, which implies that they are

analyzable under this condition, i.e., they have partial tractability. Such systems can be problematic for certain solver schemes due to the discontinuity in the ODEs introduced by the reset. Due to the fact that by the reset the whole semiplane $q_2 > 1$ is reduced to a point ($\mathbf{q} = [0, 1]$), the isochrones and radius isochrones are identical except for the point on the limit cycle. Therefore, the transformation \mathbb{T} is mathematically problematic.

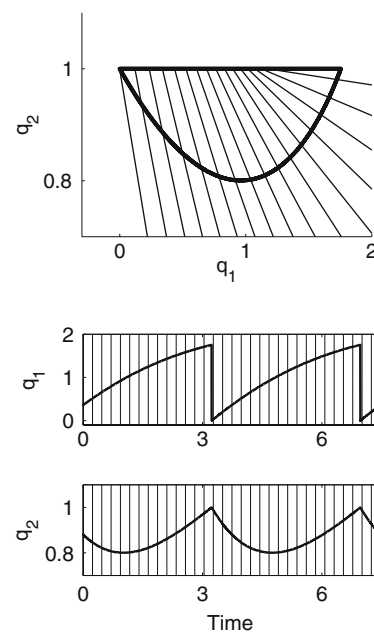
Equations From [Izhikevich \(2001\)](#)

$$\dot{q}_1 = bq_1 - \omega q_2 + I, \quad (40)$$

$$\dot{q}_2 = \omega q_1 + bq_2 \quad \text{If } q_2 > 1, [q_1, q_2] = [0, 1], \quad (41)$$

where $b < 0$ and I are constants.

Illustration



References The resonant and fire neurons in [Izhikevich \(2001\)](#).

Relaxation oscillators

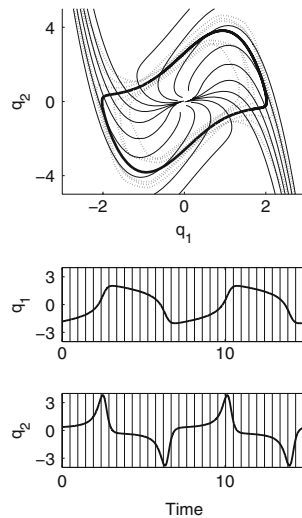
Properties In general, relaxation oscillators have no closed-form solution, characteristic phase space with sharp corners, a fast/slow system, i.e., two involved time scales that can often be related to physical mechanisms. Relaxation oscillators allow for fast phase locking due to their bent isochrones. Furthermore, they allow for naturally higher-order locking.

Equations The van der Pol oscillator

$$\dot{q}_1 = q_2 + p(t), \quad (42)$$

$$\dot{q}_2 = \mu(p^2 - q_1^2)q_2 - v^2 q_1. \quad (43)$$

Illustration



References The well-known Hodgkin–Huxley (HH) model of the giant squid axon [Hodgkin and Huxley \(1952\)](#) can exhibit relaxation-type oscillatory activity. By a simplification of the HH model, the Fitzhugh–Nagumo (FHN) oscillator [FitzHugh \(1961\)](#) was derived. The FHN is very closely related to the van der Pol oscillator.

It is important to realize that a given oscillator can belong to several of the classes presented here (i.e., isochronous oscillators, harmonic oscillators, and ACPOs are not mutually exclusive).

4 Design approaches

Thus far we have seen *what* can be designed. In this section, we address the issue of *how* design goals can be achieved. Thus, as we have seen, we need to find the structure and parameterization of either \mathbf{F} or \mathbb{T} . They can be found in different ways, with offline and online methods.

Offline methods: Here the structure and parameters are found by some process before the system is deployed. Once the system is working they remain fixed. There are different ways to find a suitable structure and parameterization of the ODEs:

- (1) The system can be designed by hand, with the help of suitable mathematical tools.
- (2) The structure can be found by search/optimization (i.e., an algorithm outside of the dynamical system).
- (3) The system can be dynamically shaped (i.e., the tuning

is part of the dynamical system), but once deployed this process is frozen.

Online adaptation: We could also imagine having some of the parameters changing over time as the system is deployed. This basically means that the parameters are no longer constant. They are turned in a certain sense into state variables as well. Again, different approaches can be used: (1) By an algorithm that is outside of the dynamical system. Usually this includes the assessment/measurement of some predefined quality of the system and an algorithm that tunes the parameters to achieve better quality. (2) Dynamically: Here the relevant parameters are turned into state variables and a dynamic law in the form of ODEs must be found that will tune the system to the required dynamics. Such an approach can be found in very recent research with oscillators (early work [Ermentrout \(1991\)](#); [Nishii \(1998, 1999\)](#), more recent [Buchli, Ijspeert \(2004\)](#); [Righetti et al. \(2006\)](#)). We only call such systems adaptive since they combine the dynamics to be exploited *and* the adaptation process into a *single* dynamical system. This is also in line with the use of the term adaptive in the framework of adaptive control but is in contrast to the use of the term in many applications of oscillators.

We can thus change the properties as discussed earlier in Sect. 3 (1a–c, 2a and b) and use the design approaches described here. In Table 3 we list some of the literature in which oscillators are used either in modeling that had an influence on robotic applications or are related to or are actual robotic applications. In the next section we will discuss some aspects of Table 3 in more detail.

5 Design results: from reactive to adaptive oscillators

Thus, we have come a long way in describing the basic characteristics of oscillators, how they lead to design specifications, and their properties. Finally, we would like to discuss some aspects of the resulting systems. The resulting systems can have different properties in terms of how they react to perturbations and how long information about perturbations is retained. The oscillator can be purely *reactive*, i.e., the perturbation has only a short-term, transient effect. Or the system can be *adaptive*. The system has a memory, and the effect of the information may last for an infinite amount of time. We note this classification in Table 3 in the first column by R/A.

Table 3 Table classifying some of the contributions of field oscillators applied to robotics and related modeling

The contributions are classified according to the design method as discussed in Sect. 4. The labels in each case correspond to the design goals identified in Sect. 3: (1) Timing: design that influences the phase of the oscillator: (a) influence on the (relative) phase, (b) instantaneous frequency, and (c) average frequency. (2) Design that influences the geometry of the oscillator: (a) influences on r and (b) the design of T^{-1} /output filter. The column “R/A” indicates whether the resulting system is reactive or adaptive. This table has sparsely populated columns that point to open research issues

contribution	R/A	Offline			Online	
		“By hand”	Algorithmic	Dynamic	Algorithmic	Dynamic
Ermentrout and Kopell (1994)	R	1a				
Williamson (1998)	R	1a				
Ijspeert et al. (2005)	R	1a				
Matsuoka (1985, 1987)	R	1a				
Schöner et al. (1990)	R	1a				
Schöner and Kelso (1988)	R	1a				
Endo et al. (2005)	R	1a				
Morimoto et al. (2006)	R	1a				
Righetti and Ijspeert (2006a)	R	1b				
Taga (1994, 1995a,b)	R	1a				
Buchli and Ijspeert (2004)	R	1a				
Fukuoka et al. (2003)	R	1a				
Schöner and Santos (2001); Santos (2003, 2004)	R	1a				
Collins and Richmond (1994)	R	1a				
Ijspeert (2001)	R		1a/2			
Zegers and Sundareshan (2003)	R		2b			
Okada et al. (2002, 2003)	R		2a			
Ijspeert et al. (2002)	R		2b			
Ruiz et al. (1998)	R		2			
Galicki et al. (1999); Leistriz et al. (2002)	R		2			
Righetti and Ijspeert (2006b)	R			1a/2b		
Marbach and Ijspeert (2005)	A				1a	
Nishii (1999)	A					1c/1a
Ermentrout (1991)	A					1c
Large (1994, 1996)	A	1a				1c
Buchli, Ijspeert (2004)	A					1c
Righetti et al. (2006); Buchli et al. (2005, 2006)						

5.1 Reactive: temporary entrainment and shape changes

The basic property of structurally stable dynamical systems can be exploited, i.e., the fact that their behavior is a combination of their intrinsic dynamics *and* external input. This means that external inputs can partially modify or even “override” the autonomous behavior of the system (annihilate attractors/induce bifurcations). This can possibly be exploited for applications. The changes to the system are reactive in the sense that there is no lasting change in the system. If the input signal is switched off, then the system will immediately behave according to its original autonomous dynamics. In other words, the parameter, i.e., intrinsic dynamics, stays constant. Memory effects can only be realized by the state of phase point and are transient and short-lived. In this category we can include all uses of synchronization, phase resetting, etc. (such as in Ermentrout and Kopell 1994; Williamson 1998; Ijspeert et al. 2005; Matsuoka 1985, 1987; Schöner et al. 1990; Schöner and Kelso 1988; Taga 1994, 1995b; Buchli and Ijspeert 2004; Fukuoka

et al. 2003; Schöner and Santos 2001; Santos 2003, 2004; Collins and Richmond 1994; Endo et al. 2005). As an example, in case of phase locking the oscillator matches its frequency to the frequency of the input. This is reactive since the frequency does not stay in the system. The system has no memory of the frequency. The very moment the input is switched off it rotates with the intrinsic frequency. The only remaining perturbation is a possible shifted phase compared to the hypothetically same but unperturbed oscillator. Thus the system is more reactive than adaptive (despite the use of the word in many contributions). One can argue that the above distinction between a reactive and an adaptive system is somewhat arbitrary, but often we can argue based on separation of time scales. The “parameters” will usually evolve on time scales slower than the “state variables.” This separation of time scales is an important concept in physics and engineering when it comes to deciding which variables are considered static and which dynamic Crawford (1991); Haken (1983).

Entrainment, synchronization, and phase locking As we saw in the previous section, the limit cycle has very

characteristic stability properties: It is marginally stable in the direction of rotation. This implies that a perturbation in this direction is not “forgotten” by the system, while the perturbations perpendicular, i.e., asymptotically stable direction, to the limit cycle are damped out.

We can exploit the synchronization properties of a limit cycle system to slightly modify an oscillator’s timing so that it works “in step” with some outside process. Meanwhile, a very common application in robotics is to exploit synchronization for legs to work in step with some sensory input (e.g., touch sensors on the foot, cf. [Fukuoka et al. 2003](#); [Taga 1994](#); [Morimoto et al. 2006](#)).

An important aspect in the design of oscillators, especially if they are coupled with others in networks, is often the question of how to design the phase relationship, i.e., with which phase lag the activity of the oscillator is synchronized with the perturbation. An interesting approach to designing specific phase lags is to use contraction theory [Wang and Slotine \(2005\)](#).

As we have seen, important concepts for the design stage are the phase response curve (PRC) [Pikovsky et al. \(2001\)](#) or the more general concept of phase sensitivity (PS) [Ermentrout and Kleinfeld \(2001\)](#); [Kramer et al. \(1984\)](#); [Buchli and Ijspeert \(2004\)](#). PRC/PS can be derived in an analytic fashion only for some types of oscillators (i.e., harmonic oscillators or phase oscillators; thus they might be the oscillator of choice for this reason). Basically having a closed form of PRC/PS is equivalent to knowing the closed form of the limit cycle. However, for other, nontractable oscillators, we can measure the PRC/PS by numerical integration.

The PRC and PS typically have zeros, which means that a perturbation arriving when the oscillator is at this phase does not affect the phase. The phase of the oscillator will thus be shifted by a perturbation until it reaches this point and remain there (provided it is a stable point). This means that to design a certain phase relationship, either we must design the PRC/PS of the system, i.e., its zeros, or we need a filter to the input making sure that the desired phase relationship is attained (e.g., rotation of signal as in [Buchli and Ijspeert 2004](#)).

A discussion about the choice of oscillators can be found in [Buchli and Ijspeert \(2004\)](#), where it is shown that the simpler oscillators allow for a good design of some of the properties of networks of oscillators used as CPGs.

A key requirement for synchronization is that the frequency of the oscillator must be close to the frequency of the input. If the difference between the frequencies is large, then the oscillator will not fully synchronize and it will only show a tendency to synchronization, an effect known as phase slips [Pikovsky et al. \(2001\)](#). On average the oscillator signals will drift relative to each

other. A way around this problem is to make the frequency adaptive, which will be discussed in the next subsection.

Another way to design a phase relationship in networks of coupled oscillators is by using the theory of symmetry, where the phase pattern can be achieved without the component oscillators satisfying necessarily the correct properties in the PRC/PS [Golubitsky and Stewart \(2006\)](#); [Schöner et al. \(1990\)](#); [Righetti and Ijspeert \(2006a\)](#). However, even with this method the consideration of the properties of the single oscillator helps to design nonfrustrated systems, i.e., systems where the individual behavior is in accordance with the global pattern; such systems settle down faster and are more stable.

The synchronization properties can also be exploited to sequence and time actions as outlined in [Schöner and Santos \(2001\)](#); [Santos \(2003, 2004\)](#).

Intercycle timing Another desired property could be that the input signal should influence the instantaneous frequency without changing the average observed frequency. The key to such behavior is property 1 in Table 2, which was discussed earlier. While we are not aware of a contribution directly exploiting this characteristic, it would be straightforward to impose an additional constraint on the oscillator used in [Righetti and Ijspeert \(2006a\)](#) to satisfy this property.

Reactive shape changes Changes of radius are less frequently exploited than entrainment effects. But of course due to stability properties we can very well imagine an input that deflects the limit cycle by changing temporarily the fixed point for the radius. This translates in the outputs into having temporarily a larger or smaller amplitude. An example can be found in [Righetti and Ijspeert \(2006b\)](#).

Shaping the dynamical system Another way of designing an oscillator with a given output is to use a system that is a universal approximator and approximate the oscillator with this dynamical system. In [Ruiz et al. \(1998\)](#); [Galicki et al. \(1999\)](#); [Leistritz et al. \(2002\)](#) the authors use recurrent neural networks to achieve arbitrary limit cycles. The disadvantage of this method is that it usually leads to a very high dimensional system in which the influence of the parameters cannot be easily grasped.

An alternative approach to changing the intrinsic dynamics of an oscillator is to shape its output into to some given form with filters. One possibility is to design the filter so that it is purely functional or to design a dynamical system that transforms the output (cf. linear filter). In [Ijspeert et al. \(2002\)](#) the authors use Gaussian filters shaped by locally weighted learning. In [Zegers and Sundareshan \(2003\)](#) the authors use neural networks as

filters. The use of filters has the disadvantage that a discontinuous change in the parameters of the filter can lead to a discontinuous change in the output (i.e., one of the advantages of using dynamical systems is negated). In Okada et al. (2003) shaping of arbitrary limit cycles is discussed using a direct design of the flow. In Righetti and Ijspeert (2006b) the authors shape the limit cycle with the help of a network of adaptive frequency oscillators.

5.2 Adaptation: lasting changes to the dynamics

In this section we address *lasting changes to the intrinsic dynamics*, i.e., adaptation. Such lasting changes could also be called learning, and in some communities this is the preferred term. We use the terms interchangeably here.

As we have seen, the limit cycle system is parameterized by a set of parameters ρ , which are usually kept constant. Adaptation means now that we find a law to change some or all of the parameters so that an adaptation goal is achieved. There are two conceptually different ways of achieving that: either the parameters are changed by an external process or algorithm or the dynamical system itself gets enhanced with additional state variables and ODEs that represent the parameters and their evolution, respectively.

This means we have to find suitable differential equations $\dot{\rho} = \mathbf{F}_{\rho}(\mathbf{q}, \rho, t)$. This in turn implies that the set of parameters reflects the state of the adaptation process at any given time, and especially also after the system is halted. We can thus possibly also read out certain information about the system. This adaptation process constitutes a longer-term memory, in contrast to the reactive changes in the previous section, which are forgotten and do not get remembered in the state of the parameters.

The first, algorithmic, method is, from the methodological point of view, simpler since standard optimization, learn, and search techniques can be used. In Marbach and Ijspeert (2005) Powell's method is used to optimize the parameters of an oscillator network. Many other optimization methods could be used for similar tasks. The second, dynamic, method is more appealing from the conceptual point of view and leads to more efficient and robust solutions (cf. Buchli et al. 2006).

Dynamic adaptation of limit cycle systems is a more recent development and a young field of research. There are some investigations on adaptation of parameters (frequency, etc.), e.g., in Nishii (1999); Large (1994); Buchli, Ijspeert (2004); Righetti et al. (2006).

An example is introduced in Buchli, Ijspeert (2004) and analyzed in detail in Righetti et al. (2006), where

a Hopf oscillator (Eqs. (30)–(31)) is enhanced with an evolution law for the frequency ω in the following way:

$$\dot{\omega} = -k \frac{y}{\sqrt{x^2 + y^2}} \cdot p(t), \quad (44)$$

where k is a coupling constant and $p(t)$ an additive perturbation to Eq. (30). This law allows the oscillator to adapt to the frequency of the perturbation $p(t)$. Such an additional law for the parameter ω endows the system with many very nice properties and can be exploited for different tasks such as adaptation to body dynamics Buchli, Ijspeert (2004); Buchli et al. (2005, 2006) or programmable CPGs Righetti and Ijspeert (2006b).

As can be seen in Table 3, the column with the adaptation is only sparsely populated. Many questions must be answered and methodologies found.

Adaptation can also be used in the design phase, and then the adaptation process is frozen for the deployment phase (i.e., online vs. offline adaptation). This means the dynamics is adapted, then remains fixed for the application (learning/exploitation phase is distinguished), whereas in the first case the adaptation works continuously.

6 Conclusion and discussion

Discussion of design choice Oscillators have been used widely in robotics over the last few years with much success; however, a lack of abstraction often leads to suboptimal solutions for a given goal. These suboptimal choices arise due to a lack of abstraction of the concepts and/or a fixation on traditionally used oscillators.

For engineering of an application with oscillators we first of all must get clear on what feature of an oscillator is the important one for the task. In other words, we must decide if it is a *generic* feature of oscillators, of a class of oscillators, or if it is unique to a certain oscillator/input/output.

We must think about readouts, what information needs to be available, e.g., an oscillatory output signal with certain properties (e.g., harmonic), or if the phase of the oscillator has to be available as output.

We can then think about whether one of the well-known oscillator/coupling schemes is suitable for the task at hand. Of course sometimes it can be helpful to take an existing oscillator and modify it to fit (cf. e.g., Kay et al. 1987; Righetti and Ijspeert 2006a).

Most of the time oscillators are used for their synchronization properties. Thus, we are interested in how the phase ϕ behaves over time. In addition, the influence on the radius can be exploited, but this is far less common. This means that for many applications the phase oscillator will be a good and completely sufficient choice.

While it is often very convenient to use a phase oscillator, we must be careful with coupling functions which explicitly contain the phase of the input signal (i.e., as is often done in work on coupled phase oscillators), because couplings that work out this way do not usually generalize so easily to a general periodic signal of which the phase is not directly accessible. One way to bridge the gap and to investigate arbitrary periodic signals is to use Fourier series.

As always in engineering it is not possible to give rules that are valid for all cases, but important questions to guide the choice of oscillator and design strategy are the following:

- First and foremost: get clear about the design goals, i.e., what property of an oscillator do you want to exploit and why?
- Is direct access to the phase or the frequency required? In other words, should they be presented as explicit variable and parameter?
- Is there a restriction on the number of state variables and complexity of integrating the system (e.g., for embedded computing)?
- Should it be possible to analytically prove or predict properties of the system (convergence, phase relationship)?
- How many elements should the system contain, i.e., is a network of oscillators needed (e.g., half-center, interneurons) or might a single oscillator be enough?
- Is there an advantage to using a strongly nonlinear oscillator or is a harmonic/phase oscillator enough, e.g., is a “neural” oscillator really the right one to achieve my goal? In modeling, does my model really concern the neural level so that the use of a neural oscillator is justified?
- Try to make the design space as orthogonal as possible (e.g., in the Matsuoka oscillator there is a strong influence on the shape if the oscillator gets coupled to others).
- Should the complexity and nonlinearity be placed in the oscillator or in the coupling (e.g., for higher-order locking: either the phase oscillator and coupling generate the higher-order frequency components or the complex oscillator and simple couplings do)?

An often made assumption in treating limit cycle systems is that the influence on phase and radius can be completely separated. This assumption directly follows from the stability directions as discussed above. However, a perturbation perpendicular to the limit cycle may in general very well have an influence on the phase and vice versa (e.g., in the van der Pol); the separation of the directions is a very useful approximation but is limited

to a region “close” to the limit cycle. If we want to have a complete separation, then the oscillator must be chosen accordingly, i.e., a harmonic isochronous oscillator.

Sometimes it is desirable to synchronize the signal in a ratio other than 1:1; in general oscillators can phase-lock in any ratio n/p $n, p \in \mathbb{N}$. However, harmonic oscillators are not sufficient to achieve this task unless the input contains higher harmonics (ideally pulselike). Relaxation oscillators can phase-lock with other ratios to a harmonic signal.

Finally, it is important to stress that there are many design aspects that could not be discussed in this article, for example, transient time for locking.

It turns out that relaxation oscillators are well suited for rapid phase locking [Somers and Kopell \(1993\)](#). This is due to the bent isochrones, i.e., even a small perturbation can drive an oscillator over many isochrones and thus advance it rapidly toward a stable phase. We see that also for such discussions, the basic principles have been presented in this paper.

We do not address another way of distinguishing two dynamical systems, namely, by their bifurcation behavior. In dynamical systems it is a typical phenomenon that if some parameters are changed beyond a critical value, the qualitative behavior of the system can completely change, e.g., an oscillator can bifurcate to fixed-point behavior. If two systems differ in their bifurcation behavior, then it is usually not possible to transform one into the other by only a change in the coordinate systems. Different oscillators can have different bifurcation behaviors. It is important to realize that we only discuss the oscillators in their oscillatory regime, far from the bifurcation points, and the above statements are only valid in this parameter range.

Furthermore, we have simplified the discussion by the fact that we concentrate only on the limit cycle attractor of the system, while it can possibly have other attractors. Thus our discussion is limited to the basin of attraction of the limit cycle attractor (e.g., every limit cycle encloses an unstable fixed point from which the solutions would not converge to the limit cycle).

Outlook and future research Classifying the design of oscillators is an immense task to, and many details could not be discussed in this article.

We are convinced that taking an engineering perspective on oscillators is needed in order to make full use of them in robotics applications. This paper is only a first step in that direction.

It is possible that an engineered system might lack some of the self-organization properties and flexibility of natural oscillatory networks, but this is a fundamental problem when trying to exploit systems capable of self-organization for engineering purposes. On the other

hand we gain methodology and we can assert guarantees for certain properties of the system, but clearly such questions are open for research; see [Buchli and Santini \(2005\)](#) for a deeper discussion.

Many substrates other than digital computers allow for structurally stable oscillators, i.e., chemical oscillators, (analog) electronic, and biological (see also [Buchli and Santini 2005](#)). Choosing such a substrate, however, narrows the degrees of freedom in the design, but still the key to understanding and engineering those systems is presented in this article. It would of course be interesting to exploit such substrates for engineering applications.

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